Differential Topology, by Milnor.

Note by Conan Leung

§ Immersion

(i.e. $M \hookrightarrow N$)

Def. $f: M^n \longrightarrow N^p$ is called (1) immersion

if $\forall\ x \in M$, $df(x): T_xM \longrightarrow T_{f(x)}N$ 1-1.

(2) embedding if moreover, f homeo. into.

Theorem. $f: M^n \longrightarrow \mathbb{R}^p$ w/ $p \geq 2n$

$\Rightarrow \exists\ g: M^n \hookrightarrow \mathbb{R}^p$ near f

immersion.

(If $Y \subset M$ s.t. $f|_Y$ has rank n, (i.e. Y is already a good set.)

then $\exists\ g$ as above & $g|_Y = f|_Y$.

Key lemma (local case): $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ w/ $p \geq 2n$

$\Rightarrow \exists\ g: \mathbb{R}^n \hookrightarrow \mathbb{R}^p$ near f

w/ $g(x) = f(x) + Ax$, $\exists$ small $A \in Mat_{p\times n}$

Pf of lemma: $Dg(x) = Df(x) + A$

$F(Q,x) \triangleq Q - Df(x): Mat^{rk \leq n-1}_{p\times n} \times \mathbb{R}^n \longrightarrow Mat_{p\times n}$

$p \geq 2n \Rightarrow \dim(\text{———} \text{"} \text{———}) < \dim(\text{—} \text{"} \text{—})$

$\Rightarrow Im(F) \subset Mat_{p\times n}$ has measure $= 0$

$\Rightarrow \exists\ A_{p\times n}$ arbitrarily close to 0

s.t. $A \neq Q - Df(x)$ $\forall\ x$, $\forall\ rk(Q) < n$

i.e. $Df(x) + A$ always $rank = n$ QED.

Pf. of thm. Pick countable locally finite cover $M = \bigcup_{i=1}^{\infty} V_i$

Modify f on V_i's one by one. Say $V_i \simeq B(3)$

$$f(x) \rightsquigarrow f(x) + \underbrace{\varphi(x)}_{\text{cutoff for } V_i} Ax.$$

$$Df(x) + (A \cdot \underbrace{x}_{\lesssim 3})\, \underbrace{d\varphi(x)}_{\lesssim c} + \underbrace{\varphi(x)}_{\lesssim 1} A$$

$\exists$ small A w/ $|Ax| < \frac{\delta}{2^i}$ & $x \in V_i \simeq B(3)$

s.t. $f + \varphi A(x)$ has max. rk. on V_i

Inductively $\rightsquigarrow$ g w/ $|g - f| < \delta$

(If $f|_Y$ has max rk. $\Rightarrow$ same for a nbd.
$\Rightarrow$ can choose V_0 to be this nbd.)

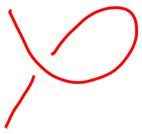
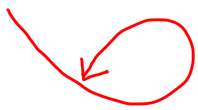
QED.

§ Embedding

<u>Theorem</u>. $\forall M^n$ $\exists$ emb. $M^n \hookrightarrow \mathbb{R}^{2n+1}$

Lemma. Given immersion $f: M^n \looparrowright \mathbb{R}^p$

$p \geq 2n+1 \implies$ 1-1 immersion $g \overset{\delta}{\sim} f$.

Remark: Emb  , 1-1 immersion 
but not emb.
($\because$ not assuming M cpt)

Pf. of lemma: As before, modify on V_i via

$f(x) \rightsquigarrow f(x) + \varphi(y)\, b$ w/ small $b \in \mathbb{R}^p$

$M \times M \dashrightarrow \mathbb{R}^p$

$(x_1, x_2) \longmapsto f(x_1) - f(x_2)$

$p > 2n \Rightarrow$ Image has measure $= 0$

Pf of theorem: 1° Construct $f_0 : M^n \longrightarrow \mathbb{R}$
w/ limit set $L(f_0) = \phi$

$L(f_0) \triangleq \{ y \in \mathbb{R} : y = \lim_{k\to\infty} f(x_k)$ $\exists$ div. seq. x_k's in $M \}$

Choose loc. finite $\bigcup_j V_j = M$ & partit$^{\text{n}}$ of 1 φ_j's

then $f_0 = \sum_j j\, \varphi_j$ works.

2° $f = (f_0, 0, \cdots 0): M^n \longrightarrow \mathbb{R}^{2n+1}$

choose 1-1 immersion $g \overset{\delta}{\sim} f \Rightarrow g$ topo. emb.

§ Transversality

<u>Theorem.</u> $f : M \longrightarrow X \overset{\text{closed}}{\supset} Y$

$\Rightarrow \quad \exists\, g : M \longrightarrow X$

close to f and $g \pitchfork Y$

(can also keep f wherever tranverse to Y already)

i.e. $\forall\, g(x) \in Y \quad \overset{\text{loc.}}{=\!=} \{h = 0\}$ w/ $h : Y \longrightarrow \mathbb{R}^{q = \mathrm{codim}(Y \subset X)}$

$\mathrm{rank}(D(h \circ g))(x) = q$

- By implicit function theorem, $f^{-1}(Y) \subset M$ is a smooth submfd (of codim q), unless ϕ.

Pf : Similar as before $f(x) \rightsquigarrow f(x) + \imath \circ (Ax + b)$

w/ loc. $\underset{M}{V_j \simeq B^n \subset \mathbb{R}^n} \xrightarrow{Ax+b} \underset{N_{Y/X}}{\mathbb{R}^q} \overset{\imath}{\hookrightarrow} \underset{X}{\mathbb{R}^m}$

§ Vector Bundles

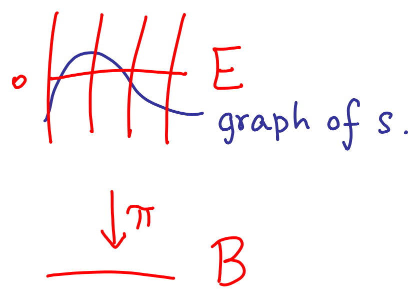
$\mathbb{R}^r \longrightarrow E \xrightarrow{\pi} B$: family of r dim vector space E_b parametrized by $b \in B$.

"smooth" family : local triviality + "smooth" transit$^{\underline{n}}$ fu.

- Linear algebra structures $\overset{\text{family}}{\Longrightarrow}$ VB str.

Sections ($s_b \in E_b \quad \forall\, b \in B$

$\Leftrightarrow$ $s : B \to E \quad \pi \circ s = 1_B$)



$\oplus$, $\otimes$, pullback.

Inner product h ($\exists$ via partition of unity)

- Tangent bundle TM

$$\begin{matrix} \mathbb{R}^N \\ \cup \\ M \end{matrix} \quad \Rightarrow \quad \begin{matrix} T\mathbb{R}^N = \coprod_{x \in \mathbb{R}^N} \mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N \\ \cup \\ TM = \coprod_{x \in M} T_xM \end{matrix}$$

$T_xM \ni v$ has an intrinsic characterisation:

1) 1st order part of a curve $\gamma(t)$ thru. x

i.e. $v = \gamma'(0)$ w/ $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ & $\gamma(0) = x$

2) differentiation of function f along γ at x.

i.e. $v : C^\infty(M) \longrightarrow \mathbb{R}$

$v(f) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t))$

In particular, $v(f\,g) = v(f)\, g(x) + f(x)\, v(g)$

3) In loc. coord $(u^1, \cdots, u^n)$ around x, $v = \sum a^i \frac{\partial}{\partial u^i}$.

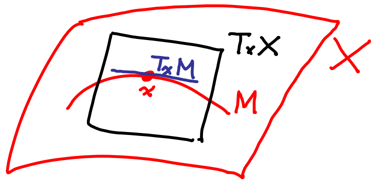
- $F : M \longrightarrow X$

$$dF(x) : T_x M \longrightarrow T_{f(x)} X$$

$$(dF(x)(v))(f) := v(f \circ F)(x).$$

So, $dF \in \Gamma(M, T^*M \otimes F^*TX) = \Omega^1(M, F^*TX)$

- Embedding $F : M \hookrightarrow X$



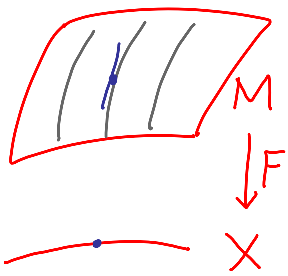
$\Rightarrow \quad T_M \leqslant F^*T_X =: T_X|_M$

$\rightsquigarrow$ normal bundle $N_{M/X}$ as quotient bundle

i.e. $\exists$ short exact seq. of VB / M

$$0 \longrightarrow T_M \longrightarrow T_X|_M \longrightarrow N_{M/X} \longrightarrow 0$$

- Submersion $F : M \twoheadrightarrow X$



i.e. $\mathrm{rk} F \overset{\text{everywhere}}{=\!=\!=} \dim X$

i.e. $T_M \twoheadrightarrow F^*T_X$

$\rightsquigarrow$ vertical tangent bundle $T_{M/X}$ as kernel bundle

i.e. $\quad 0 \longrightarrow T_{M/X} \longrightarrow T_M \longrightarrow F^*T_X \longrightarrow 0$

Theorem: $\mathbb{R}^r \longrightarrow E \xrightarrow{\pi} B$

B compact $\Longrightarrow \exists$ VB F/B s.t. $E \oplus F \simeq \underline{\mathbb{R}^N}$

Proof: B compact

$\Rightarrow$ finite cover $U_1 \cup \cdots \cup U_k = B$

s.t. $\forall j$, $E|_{U_j}$ trivial, i.e. $E|_{U_j} \xrightarrow[\cong]{f_j = (f_j^1, \ldots, f_j^r)} U_j \times \mathbb{R}^r$

Let $(\varphi_j : U_j \to \mathbb{R})$'s partition of unity

Define $E \longrightarrow B \times \mathbb{R}^{rk}$ inj. homo. $\Rightarrow \checkmark$

$e \longmapsto \left(b = \pi(e), \left(\varphi_j(b) f_j^i(e)\right)_{\substack{i \le r \\ j \le k}} \right)$

Def: VB $E, F/B$ stably equivalent (s-eq.)

if $E \oplus \underline{\mathbb{R}^{n_1}} \simeq F \oplus \underline{\mathbb{R}^{n_2}}$ $\quad \exists\, n_1, n_2$

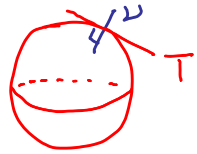
Cor. $E_0 - E_1$ is well-def^d for s-eq. classes of VB/B

(i.e. $E_1 \oplus F \overset{\text{Thm}}{\simeq} \underline{\mathbb{R}^N} \Rightarrow [-E_1] = [F]$)

Namely, $\{ [\underset{\in \text{VB}/B}{E}], \oplus \}$ is Abelian group.

Ex: S-eq. class of $\nu_{M/\mathbb{R}^N}$ is indep. of immersion.

Ex: TS^2 is stably trivial.



$(\because\ TS^2 \oplus \nu_{S^2/\mathbb{R}^3} = \underline{\mathbb{R}}^3).$

Prop. T_M s-trivial $\iff \exists\, M \hookrightarrow \mathbb{R}^N$ w/ $\nu_{M/\mathbb{R}^N}$ trivial

Pf. [$\Leftarrow$] trivial.

[$\Rightarrow$] $\quad M \hookrightarrow \mathbb{R}^N$

$$\Rightarrow \quad T_M \oplus \nu_{M/\mathbb{R}^N} \cong \underline{\mathbb{R}}^N$$

s-trivial $\qquad$ trivial

s-trivial

$$\text{i.e.} \quad \nu_{M/\mathbb{R}^N} \oplus \underline{\mathbb{R}}^q \simeq \underline{\mathbb{R}}^{N-n+q}$$

$$\Rightarrow \quad M \hookrightarrow \mathbb{R}^N \times 0 \subset \mathbb{R}^{N+q}$$

has normal bdl. $\nu_{M/\mathbb{R}^{N+q}} = \nu_{M/\mathbb{R}^N} \oplus \underline{\mathbb{R}}^q \simeq \underline{\mathbb{R}}^{N-n+q}$.

QED.

§ Thom's cobordism theory

Def. $M_1^n \sim M_2^n$ cobordant

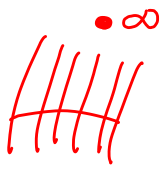
if $M_1 \sqcup M_2 = \partial Q$ $\exists Q^{n+1}$

- $\Omega^n \triangleq \{M^n, \sqcup\}/\sim$ is Abelian group w/ id. $= S^n$ and every element has order 2.
- $\times : \Omega^{n_1} \times \Omega^{n_2} \xrightarrow[\text{mfd.}]{\text{product}} \Omega^{n_1+n_2}$ (well-def^d ✓)
- $\Omega \triangleq \bigoplus_{n=0}^{\infty} \Omega^n$, $\sqcup$, $\times$ $\rightsquigarrow$ graded comm. ring w/ 1
  graded alg. / $\mathbb{Z}_2$.

<u>Thom's theorem</u>: $\Omega = \mathbb{Z}[X_1, X_2, \ldots]/\sim$

$\dim X_n = n$ and $n \neq 2^m - 1$ and $X_{2m} = \mathbb{RP}^{2m}$.

Thom space for $\mathbb{R}^r \to E \to B$



$\underline{\underline{\text{def.}}}$ $\mathcal{T}(E) := E \cup \infty$

$= E / E_{\geqslant 1}$ ← all $e \in E$ w/ $|e| \geqslant 1$ wrt $(E, |\cdot|)$. (if B compact)

$= E_{\leqslant 1} / E_{=1}$

(Similar to $S^r = \mathbb{R}^r \cup \infty = \mathbb{R}^r/\{|x| \geqslant 1\} = D^r/S^r$)

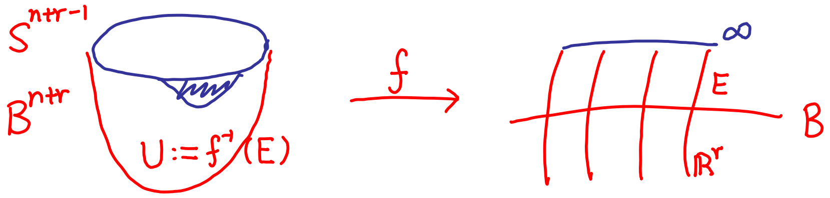
Eg. $M \subset \mathbb{R}^N \Rightarrow \mathcal{T}(\nu_{M/\mathbb{R}^N}) = \dfrac{\mathbb{R}^N}{\mathbb{R}^N \setminus \text{nbd. of } M.}$

all outside as 1 pt.

- Construction

$\forall\ [f] \in \pi_{n+r}(\mathcal{T}(E), \infty)$

$\rightsquigarrow\ f : (B^{n+r}, S^{n+r-1}) \longrightarrow (\mathcal{T}(E), \infty)$



Approx. by smooth map $f : U \longrightarrow E$

transverse to $B \subset E$

$\Rightarrow\ M^n := f^{-1}(B)$ compact submfd. in U

(indep. of $\dim B$)

<u>Theorem.</u> Given $\mathbb{R}^r \to E \to B$, then above construction gives well-def^{d} homomorphism

$$\lambda : \pi_{n+r}(\mathcal{T}(E), \infty) \longrightarrow \Omega^n$$

<u>Theorem.</u> For the universal bundle

$$\mathbb{R}^r \longrightarrow \mathcal{E} \longrightarrow Gr(r, r+m)$$

$$\lambda : \pi_{n+r}(\mathcal{T}(\mathcal{E}), \infty) \longrightarrow \Omega^n$$

(1) λ onto if $r-1, m \geqslant n$

(2) λ 1-1 if $r-1, m \geqslant n+1$.

Pf of (1): $\forall\, M^n \Rightarrow \exists\, M^n \hookrightarrow \mathbb{R}^{n+r} \simeq B^{n+r}$ if $r \geqslant n+1$

Gauss map: $G : M \longrightarrow Gr(r, r+n) \hookrightarrow \underset{m \geqslant n}{Gr(r, r+m)}$

$$G(x) = \nu_{M/B^{n+r}, x} \subset \mathbb{R}^{n+r}$$

then $\nu_{M/B^{n+r}} = G^*(\mathcal{E})$

$$M \subset \nu_{M/\mathbb{R}^{n+r}} \simeq nbd_{\leqslant 2}(M) \subset B^{n+r}$$

$$\rightsquigarrow \quad f : \overline{B}^{n+r} \longrightarrow \frac{\overline{B}^{n+k}}{B \setminus \nu_{\leqslant 1}} \simeq \mathcal{T}(\nu_{M/B^{n+r}}) \longrightarrow \mathcal{T}(\mathcal{E})$$

$$f(\partial \overline{B}^{n+r}) \;=\!=\!=\; \infty \quad = \quad \infty$$

$$M \;=\!=\!=\!=\; f^{-1}(Gr)$$

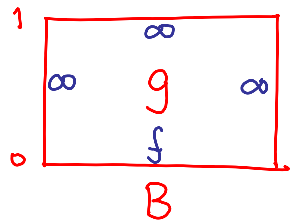
$\rightsquigarrow [f] \in \pi_{n+r}(\mathcal{T}(\mathcal{E}), \infty)$ s.t. $\lambda[f] = [M]$

Hence, λ onto.

Pf. of (2). Given $f : (B^{n+r}, \partial B^{n+r}) \longrightarrow (\mathcal{T}(\mathcal{E}), \infty)$, $f \pitchfork Gr$

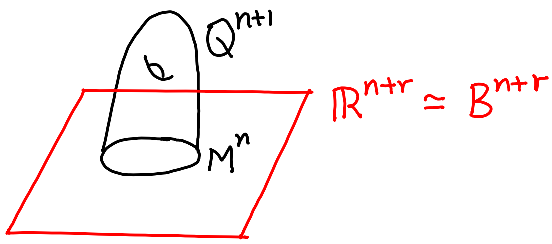
Assume $M^n := f^{-1}(Gr) = \partial Q^{n+1}$

$\overset{?}{\Rightarrow} \exists\, g : B^{n+r} \times I \to \mathcal{T}(\mathcal{E})$ s.t.



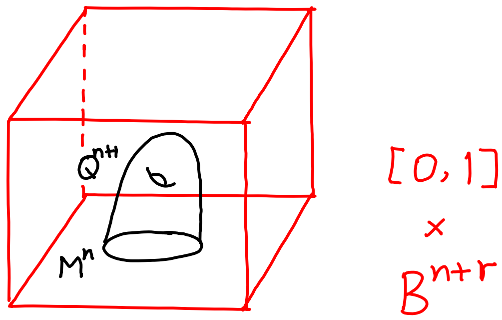
and $Q = g^{-1}(Gr)$.

Given



$r \geqslant n+2 \Rightarrow \exists$ embed $Q^{n+1} \subset \mathbb{R}^{2(n+1)+1} \subset \mathbb{R}^{n+r} \times \mathbb{R}$

More precisely,



Using obstruction theory of Steenrod

$\exists\ g_0 : \mathrm{nbd}(Q \subset B \times I) \longrightarrow \mathcal{T}(\mathcal{E})$

extend f and $g|_{\mathrm{nbd} \setminus Q} \subset \mathcal{E} \setminus Gr$

$\rightsquigarrow$ g as above. QED.

- Given $\mathbb{R}^k \longrightarrow E \longrightarrow B$

$\forall n$, $\varphi_n : \pi_{n+k}(\mathcal{T}(E)) \longrightarrow H_n(B, \mathbb{Z})$

If $n < k-1 \Rightarrow \varphi_n$ isom. (Thom isomorphism)

Need to compute $H_n(Gr(k, m+k), \mathbb{Z})$ w/ $k > n+1$.

<u>Theorem</u> (Thom) $\Omega_*^{ori} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \cdots]$

In particular, $\mathrm{rank}\ \Omega_{4k}^{ori} = \#$ partition of k

Given M smooth compact oriented mfd.,

$kM = \partial X$ $\exists k$ $\exists$ oriented X

$\Longleftrightarrow$ All Pontrjagin numbers of M vanish.

<u>Result</u> of Wall: In un-oriented cases

$M = \partial X$

$\Longleftrightarrow \begin{cases} \text{All Pontrjagin numbers of } M \text{ vanish.} \\ \text{All Stiefel-Whitney numbers of } M \text{ vanish.} \end{cases}$